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On a Property of Bases in Banach and Hilbert Spaces

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ABSTRACT

In this work we consider the problem of conservation of the bases' in Banach spaces after small perturbations for the purpose of applying obtained results to investigation of spectral expansions associated with differential operators. Having known asymptotics of the system of Eigen and adjoin functions for differential or pseudodifferential operators, it is possible to indicate its basisness in Banach space.

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INTRODUCTION

During the second half of the XX-th and early XXI centuries bases in Banach spaces attracted a lot of interest. The one of the reasons is its connection with solutions of the problems of heat conductivity and the theory of oscillations. N.K. Bary placed for consideration a class of bases in the Hilbert space H and called them the Riesz bases. Grothendieck (1955) investigated Banach spaces that have approximation property. Enflo (1972) constructed reflexive separable Banach space, where basis does not exist.

Kostyuchenko (1953) demonstrated Riesz`s basisness in $L_2[0, \pi]$ of the system of eigenvectors of Sturm-Louisville operator. Ilyin (1986) started to study necessary and sufficient conditions for Riesz`s basisness of the system of eigen and adjoin functions for ordinary differential operators of any order. Ilyin and Moiseev (1994) studied necessary and sufficient conditions for the Riesz baseness of root vectors of the second order discontinued operators.

PROOF OF THE MAIN RESULT

Recall that a sequence $\varphi = \{\varphi_n\}_1^\infty$ is called a *basis* of the Banach space E if any vector $x \in E$ is expanded uniquely into the series

$$x = \sum_{n=1}^{\infty} c_n \varphi_n \tag{1}$$

which is convergent in the norm of the space E. We exclude from consideration Banach spaces not possessing the property of approximation (see Grothendieck 1955) and Enflo (1973)).

The coefficients c_n in the expansion (1) are linear functionals:

$$c_n = f_n(x) \quad (n = 1, 2, 3, \ldots) \tag{2}$$

and according to well-known Banach theorem (see, for example, [3]), there exists a constant C_φ so that

$$\|\varphi_n\|^{-1} \le \|f_n\| \le C_\varphi \|\varphi_n\|^{-1}. \tag{3}$$

A sequence $\{\psi_n\}_1^\infty$ of elements of the Banach space E is called *a basis with a finite defect* if co $\dim L < \infty$, where $L = \overline{span\{\psi_n\}_1^\infty}$ is a linear closed envelope of the sequence $\{\psi_n\}_1^\infty$, $E = L \dotplus M$, $\operatorname{codim} L = \dim M$ and $\{\psi_n\}_1^\infty$ is a basis in L.

If there is a proper subsequence of vectors $\{\psi_{n_k}\}_1^\infty$ which forms a basis with a finite defect, then the system $\{\psi_n\}_1^\infty$ is said to be *an overfilled basis with a finite defect in E.*

A sequence $\psi = \{\psi_n\}_1^\infty$ of vectors from E is said to be ω*-linearly independent* if the equality

$$\sum_{n=1}^{\infty} c_n \psi_n = 0$$

is impossible for

$$\sum_{n=1}^{\infty} |c_n|^2 \cdot \|\psi_n\|^2 > 0 . \tag{4}$$

The following statement is valid.

Theorem 1. *Let $\{\varphi_n\}_1^\infty$ be a normalized basis in E and the system $\{\psi_n\}_1^\infty$ satisfies the condition*

$$\sum_{n=1}^{\infty} \|\varphi_n - \psi_n\| < \infty .$$

Then the system $\{\psi_n\}_1^\infty$ is a basis (perhaps, overfilled) with a finite defect in the Banach space E.

Proof. For any $\varepsilon > 0$, there exists a natural number $N = N(\varepsilon)$ such that

$$\sum_{n=N+1}^{\infty} \|\varphi_n - \psi_n\| < \varepsilon .$$

Choosing $\varepsilon = \dfrac{1}{2C_\varphi}$, where $C_\varphi = \sup\limits_n \|f_n\|$, we find N such, that

$$\sum_{n=N+1}^{\infty} \|\varphi_n - \psi_n\| < \frac{1}{2C_\varphi} .$$

Denote

$$\tilde{\psi}_n = \begin{cases} \varphi_n & \text{if } n \le N, \\ \psi_n & \text{if } n > N. \end{cases}$$

It is sufficient to show that the system $\{\tilde{\psi}_n\}_1^\infty$ is a basis. It is well known that for

$$x = \sum_{n=1}^{\infty} c_n \varphi_n$$

we have $c_n = f_n(x)$, where $f_n \quad (n = 1, 2, 3, \dots)$ are linear continuous functionals.

Introduce an operator

$$Sx = \sum_{n=1}^{\infty} f_n(x)(\varphi_n - \tilde{\psi}_n)$$

(see Kreyn (1940)). This operator is defined for each $x \in E$ and is linear bounded with the norm

$$\|S\| = \sup_{x \in E} \frac{\|Sx\|}{x} \le \frac{1}{2},$$

since

$$\|Sx\| \le \sum_{n=1}^{\infty} \|f_n\| \, \|x\| \, \|\varphi_n - \tilde{\psi}_n\| \le C_\varphi \|x\| \sum_{n=N+1}^{\infty} \|\varphi_n - \psi_n\| < \frac{1}{2}\|x\|.$$

Thus, if we determine an operator U by the equality

$$Ux = x - Sx = \sum_{n=1}^{\infty} f_n(x)\tilde{\psi}_n,$$

then U has the inverse operator $U^{-1} = (I - S)^{-1} = I + S + S^2 + \dots$.

Substitute $y = U^{-1}x$ in to the equality

$$y = \sum_{n=1}^{\infty} f_n(y)\varphi_n .$$

Then we obtain

$$U^{-1}x = \sum_{n=1}^{\infty} f_n\left(U^{-1}x\right)\varphi_n \ ,$$

act on the both parts of the last equality by the operator *U*. Taking into account the equality $U\varphi_n = \tilde{\psi}_n \quad (n = 1, 2, 3, \ldots)$ we get

$$x = \sum_{n=1}^{\infty} f_n\left(U^{-1}x\right)\tilde{\psi}_n \ .$$

The expansion

$$x = \sum_{n=1}^{\infty} d_n \tilde{\psi}_n \tag{5}$$

is unique, i.e. $d_n = f_n\left(U^{-1}x\right) \quad (n = 1, 2, 3, \ldots)$. It results in direct application of the operator U^{-1} to the both parts of equality (5). Hence the system $\{\tilde{\psi}_n\}$ forms a base in *E*.

Theorem 2. *Let* $\{\varphi_n\}_1^{\infty}$ *be a normalized basis in E. If the system* $\{\psi_n\}_1^{\infty}$ *is* ω *-linearly independent and the following condition*

$$\sum_{n=1}^{\infty} \|\varphi_n - \psi_n\| < \infty$$

holds, then the system $\{\psi_n\}_1^{\infty}$ *also forms a base in the Banach space E.*

Proof. Fix a natural number *N* and denote

$$\tilde{\psi}_n = \begin{cases} \varphi_n \text{ if } n \le N, \\ \psi_n \text{ if } n > N. \end{cases}$$

Introduce operator $S : E \to E$ which maps an element

$$x = \sum_{n=1}^{\infty} f_n(x)\varphi_n$$

to the element

$$Sx = \sum_{n=1}^{\infty} f_n(x)(\varphi_n - \tilde{\psi}_n).$$

It is obvious,

$$\|Sx\| \le C_\varphi \|x\| \sum_{k=N+1}^{\infty} \|\varphi_n - \psi_n\| < \varepsilon \|x\|$$

for sufficiently large N. Hence, there exists a linear operator U^{-1} which is inverse to the operator

$$Ux = x - Sx = \sum_{n=1}^{\infty} f_n(x)\tilde{\psi}_n .$$

Acting on the both parts of the equality

$$U^{-1}x = \sum_{n=1}^{\infty} f_n(U^{-1}x)\varphi_n$$

by the operator U, we obtain

$$x = \sum_{n=1}^{\infty} f_n(U^{-1}x)\tilde{\psi}_n . \tag{6}$$

Hence, the system $\{\tilde{\psi}_n\}_1^{\infty}$ forms a basis in E.

As the system $\{\tilde{\psi}_n\}_1^{\infty}$ forms a basis in E, then

$$\psi_k = \sum_{n=1}^{N} f_n(U^{-1}\psi_k)\varphi_n + \sum_{n=N+1}^{\infty} f_n(U^{-1}\psi_k)\psi_n = x_k^1 + x_k^2 ,$$

where $k = 1, 2, 3, \ldots, N$,

$$x_k^1 = \sum_{n=1}^{N} f_n(U^{-1}\psi_k)\varphi_n , \quad x_k^2 = \sum_{n=N+1}^{\infty} f_n(U^{-1}\psi_k)\psi_n .$$

The α-linearly independence of the system $\{\psi_n\}_1^\infty$ implies the linear independence of the system $\{x_k^1\}_1^N$ too. Indeed, if

$$\sum_{k=1}^{N}\alpha_k x_k^1 = 0,$$

then

$$\sum_{k=1}^{N}\alpha_k\cdot\left(\psi_k - x_k^2\right)=0\,.$$

Hence we have

$$\sum_{k=1}^{N}\alpha_k\psi_k - \sum_{n=N+1}^{\infty}\left(\sum_{k=1}^{N}\alpha_k f_n\left(U^{-1}\psi_k\right)\right)\psi_n = 0\,.$$

α-linearly independence of the system $\{\psi_n\}_1^\infty$ implies $\alpha_k = 0$ where $k = 1, 2, 3, \ldots, N$. Since the linear independence and baseness are equivalent for a finite-dimensional space, then

$$\varphi_n = \sum_{k=1}^{N} a_{nk}x_k^1$$

at $n = 1, 2, 3, \ldots, N$.

Hence we have, that

$$\begin{aligned}
x &= \sum_{n=1}^{N} f_n\left(U^{-1}x\right)\varphi_n + \sum_{n=N+1}^{\infty} f_n\left(U^{-1}x\right)\psi_n = \sum_{n=1}^{N}\left(f_n\left(U^{-1}x\right)\sum_{k=1}^{N}a_{nk}x_k^1\right)+\\
&+\sum_{n=N+1}^{\infty} f_n\left(U^{-1}x\right)\psi_n = \sum_{n=1}^{N}\sum_{k=1}^{N} f_n\left(U^{-1}x\right)a_{nk}\left(\psi_k - x_k^2\right) + \sum_{n=N+1}^{\infty} f_n\left(U^{-1}x\right)\psi_n =\\
&= \sum_{k=1}^{N}\left(\sum_{n=1}^{N} f_n\left(U^{-1}x\right)a_{nk}\right)\psi_k - \sum_{k=1}^{N}\left(\sum_{n=1}^{N} f_n\left(U^{-1}x\right)a_{nk}\right)x_k^2 + \sum_{n=N+1}^{\infty} f_n\left(U^{-1}x\right)\psi_n =\\
&= \sum_{k=1}^{N}\left(\sum_{n=1}^{N} f_n\left(U^{-1}x\right)a_{nk}\right)\psi_k - \sum_{k=1}^{N}\left(\left(\sum_{n=1}^{N} f_n\left(U^{-1}x\right)a_{nk}\right)\sum_{m=N+1}^{\infty} f_m\left(U^{-1}\psi_k\right)\psi_m\right)+\\
&+\sum_{n=N+1}^{\infty} f_n\left(U^{-1}x\right)\psi_n = \sum_{k=1}^{N}\left(\sum_{n=1}^{N} f_n\left(U^{-1}x\right)a_{nk}\right)\psi_k +\\
&+\sum_{m=N+1}^{\infty}\left(-\sum_{k=1}^{N}\left(\sum_{n=1}^{N} f_n\left(U^{-1}x\right)a_{nk}\right) f_m\left(U^{-1}\psi_k\right)\right)\psi_m + \sum_{n=N+1}^{\infty} f_n\left(U^{-1}x\right)\psi_n.
\end{aligned}$$

Thus, the following equality

$$x=\sum_{k=1}^{N}\left(\sum_{n=1}^{N}f_n\left(U^{-1}x\right)a_{nk}\right)\psi_k+\sum_{m=N+1}^{\infty}\left(f_m\left(U^{-1}x\right)-\sum_{k=1}^{N}\left(\sum_{n=1}^{N}f_n\left(U^{-1}x\right)a_{nk}\right)f_m\left(U^{-1}\psi_k\right)\right)\psi_m$$

is valid. It means that the system $\{\psi_n\}_1^\infty$ is a basis in the Banach space *E.*

In Bary (1951) placed for consideration a class of bases in the Hilbert space *H* and called them the Riesz bases. It is a system $\{\varphi_n\}_1^\infty$ that is the Riesz basis if there is a reversible bounded linear transformation *A* of the Hilbert space *H* onto *H*, which transforms the system $\{\varphi_n\}_1^\infty$ into an orthonormal basis. It should be noted that the problem of Riesz bases of the system of root vectors for differential operators was studied in Ilyin (1986) and Ilyin *et al.* (1994).

We say that a system $\{\psi_n\}_1^\infty$ of elements of a Hilbert space *H* is the *Riesz basis with a finite defect* if $\operatorname{codim} L<\infty$, where $L=\overline{span\{\psi_n\}_1^\infty}$ and $\{\psi_n\}_1^\infty$ is the Riesz basis in *L*. If there is a proper subsequence of vectors $\{\psi_{n_k}\}_1^\infty$ forming the Riesz basis with a finite defect, then the system $\{\psi_n\}_1^\infty$ is said to be *overfilled Riesz basis with a finite defect in H.*

Theorem 3. *Let the system* $\{\varphi_n\}_1^\infty$ *forms the Riesz basis in a Hilbert space H. Let* $\{\psi_n\}_1^\infty$ *is an almost normalized system, quadratically close to the system* $\{\varphi_n\}_1^\infty$, *i.e. it satisfies the condition*

$$\sum_{n=1}^{\infty}\left\|\varphi_n-\psi_n\right\|^2<\infty. \tag{7}$$

Then the system $\{\psi_n\}_1^\infty$ *is the Riesz basis (perhaps overfilled) with a finite defect in the Hilbert space H.*

Proof. Let the system $\{\varphi_n\}_1^\infty$ form the Riesz basis in a Hilbert space *H*. Then the sequence $\{\psi_n\}_1^\infty$ is complete in *H* and there exist constants $a_1, a_2 > 0$ such, that the inequality

$$a_1 \sum_{j=1}^{n} \left|\nu_j\right|^2 \le \left\| \sum_{j=1}^{n} \nu_j \varphi_j \right\|^2 \le a_2 \sum_{j=1}^{n} \left|\nu_j\right|^2 \tag{8}$$

holds for any natural *n* and all complex numbers $\nu_1, \nu_2, \nu_3, \ldots, \nu_n$.

For any $\varepsilon > 0$, there exists a natural number $N = N(\varepsilon)$ such, that

$$\sum_{n=N+1}^{\infty} \left\|\varphi_n - \psi_n\right\|^2 < \varepsilon .$$

Let $\varepsilon < a_1$. We construct the following system $\{\tilde{\psi}_n\}_1^\infty$:

$$\tilde{\psi}_n = \begin{cases} \varphi_n \text{ if } n \le N, \\ \psi_n \text{ if } n > N. \end{cases}$$

Note that the system $\{\tilde{\psi}_n\}_1^\infty$ is the Riesz basis. In fact,

$$\sum_{n=1}^{\infty} \left\|\varphi_n - \tilde{\psi}_n\right\|^2 = \sum_{n=N+1}^{\infty} \left\|\varphi_n - \psi_n\right\|^2 < \varepsilon .$$

The system $\{\tilde{\psi}_n\}_1^\infty$ is ω-linearly independent. Assume that it is not the case. Then there exists a sequence of complex numbers $\{c_n\}_1^\infty$ such, that

$$0 < \sum_{n=1}^{\infty} \left|c_n\right|^2 < \infty$$

and $\sum_{n=1}^{\infty} c_n \tilde{\psi}_n = 0$. This implies

$$a_1\sum_{n=1}^{\infty}|c_n|^2 \le \left\|\sum_{n=1}^{\infty}c_n\varphi_n\right\|^2 = \left\|\sum_{n=1}^{\infty}c_n\varphi_n - \sum_{n=1}^{\infty}c_n\tilde{\psi}_n\right\|^2 = \left\|\sum_{n=N+1}^{\infty}c_n\left(\varphi_n-\psi_n\right)\right\|^2 \le$$

$$\le\left[\sum_{n=N+1}^{\infty}|c_n|\left\|\varphi_n-\psi_n\right\|\right]^2 \le \sum_{n=N+1}^{\infty}|c_n|^2\sum_{n=N+1}^{\infty}\left\|\varphi_n-\psi_n\right\|^2 < \varepsilon\sum_{n=1}^{\infty}|c_n|^2 ,$$

i.e.

$$a_1\sum_{n=1}^{\infty}|c_n|^2 < \varepsilon\sum_{n=1}^{\infty}|c_n|^2,\ 0<\varepsilon<a_1 ,$$

and this inequality is impossible. This means that $\{\tilde{\psi}_n\}$ is linearly independent. By theorem of Bary (1951), the system $\{\tilde{\psi}_n\}$ is the Riesz basis.

It follows from the definition of $\tilde{\psi}_n$, that the system $\{\psi_n\}_{N+1}^{\infty}$ forms the Riesz basis with a finite defect (more exactly, the dimension of the defect subspace is equal to *N*).

The more so, because the system $\{\psi_n\}_1^{\infty}$ is the Riesz basis (may be overfilled) with a finite defect in Hilbert space H.

Remark. For any complete orthonormal system $\{\varphi_n\}_1^{\infty}$ in *H* there is an orthonormal system $\{\psi_n\}_1^{\infty}$ such that for any $\varepsilon>0$ the condition

$$\sum_{n=1}^{\infty}\left\|\varphi_n-\psi_n\right\|^{2+\varepsilon}<\infty$$

holds and, at the same time, the system $\{\psi_n\}_1^{\infty}$ is not complete.

The validity of this remark follows immediately from the following result by N.K. Bary (1951):

If numbers ρ_n $\left(0 \le \rho_n \le \sqrt{2}\right)$ satisfy the following condition that the series $\sum_{n=1}^{\infty} \rho_n^2$ diverges, then for any complete orthonormal system $\{\varphi_n\}_1^\infty$ there is an incomplete orthonormal system $\{\psi_n\}_1^\infty$, so that $\|\varphi_n - \psi_n\| = \rho_n$ $(n = 1, 2, 3, \dots)$.

Hence, we may set $\rho_n = \frac{1}{\sqrt{n}}$. Then, according to the result by N.K.Bary, for any orthonormal system $\{\varphi_n\}_1^\infty$, there is such an incomplete orthonormal system $\{\psi_n\}_1^\infty$, so that $\|\varphi_n - \psi_n\| = \frac{1}{\sqrt{n}}$.

Thus, the exact index of closeness to the Riesz basis is equal to 2.

CONCLUSIONS

Generally, the methods of Functional Analysis were employed to proof main theorems. Basisness in Banach space for ω-linearly independent system under small perturbations is robust. Having known main parts of asymptotics of the system of eigen and adjoin vectors for eliptic operators, it is possible to indicate its basisness in Banach space.

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REFERENCES

Bary, N. 1951. Biorthogonal systems and bases in a Hilbert space. *Uchenie zapiski MGU.* **4**(148): 69-107 (in Russian).

Enflo, P. 1973. A counterexample to the approximation problem in Banach spaces. *Acta Math.* **130**(3-4): 309-317.

Gokhberg and Kreyn, M. 1965. Introduction to the Theory of Linear Not-self adjoint Operators. Moscow. *Nauka:* 448 (in Russian).

Grothendieck. 1955. Produits tensoriels topologiques et espaces nucleaires. *Mem. Amer. Math. Soc.* **16**: 13-30.

Ilyin, V. 1986. Necessary and sufficient conditions for the Riesz baseness of root vectors of the second order discontinued operators, *Diff. uravneniya*. **22**(12): 2059-2071 (in Russian).

Ilyin, V. and Moiseev, E. 1994. On the absence of the baseness property of the system of root functions of the problem with an inclined derivative. *Diff. Uravneniya*. **30**(1): 128-143 (in Russian).

Kreyn, M., Milman, D. and Rutman, M. 1940. On a property of the basis in Banach spaces. *Zap. Hark. Math.* **4**(16): 106-110 (in Russian).